Q1] The following question has 2 parts.

a): For what values of a, b, is $F(x, y) = axy\hat{\imath} + (x^2 + by)\hat{\jmath}$ conservative? (5 points)

↳ Here, $P = axy$ and $Q = x^2 + by$.

To be conservative, $\nabla \times \vec{F} = 0$, but for 2-D, just $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$

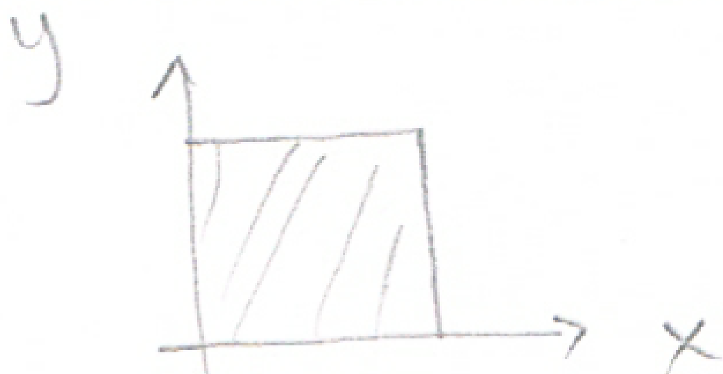
(ie. $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$)

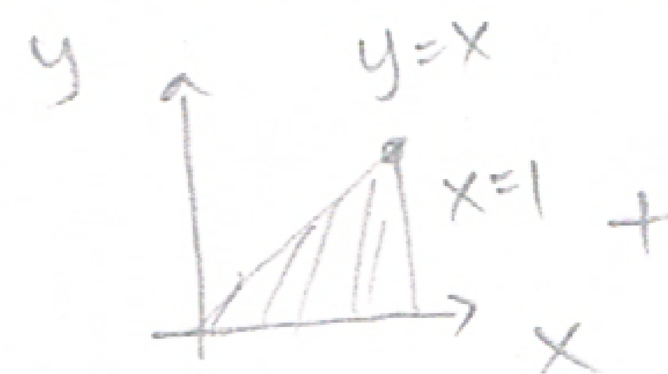
- $\frac{\partial Q}{\partial x} = 2x$
- $\frac{\partial P}{\partial y} = ax$. $\Rightarrow$ For these to be equal,

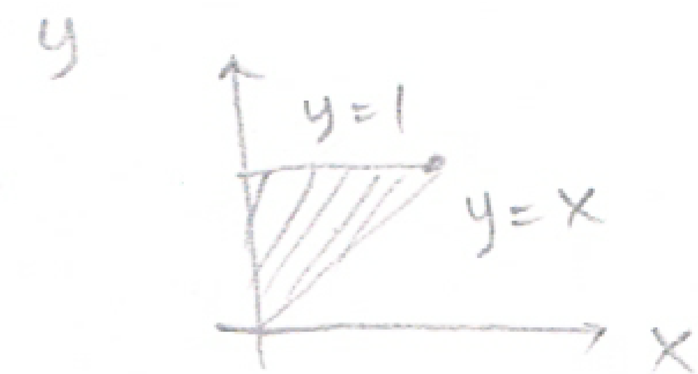
We need $\boxed{a = 2}$ and

$\boxed{b \text{ can be anything}}$.

b): Rewrite the integral $\int_0^1 \int_0^1 f dydx$ to polar form $\int \int dr d\theta$. (5 points)

Here, 

has to split as 

+ 

in order to do polar. Then, we see $x = 1$ $y = 1$

boundaries become respectively, $r\cos\theta = 1$ $r\sin\theta = 1$.

with θ bounds from line $y = x$ ie. $\begin{cases} r = \sec\theta \\ 0 \le \theta \le \pi/4 \end{cases}$ $\begin{cases} r = \csc\theta. \\ \frac{\pi}{4} \le \theta \le \pi/2 \end{cases}$

Thus, $\int_0^1 \int_0^1 f(x,y)\, dy\, dx$

$$= \int_{\theta=0}^{\pi/4} \int_{r=0}^{\sec\theta} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta + \int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{r=0}^{\csc\theta} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Q2] The following question has 2 parts. (5 points each)

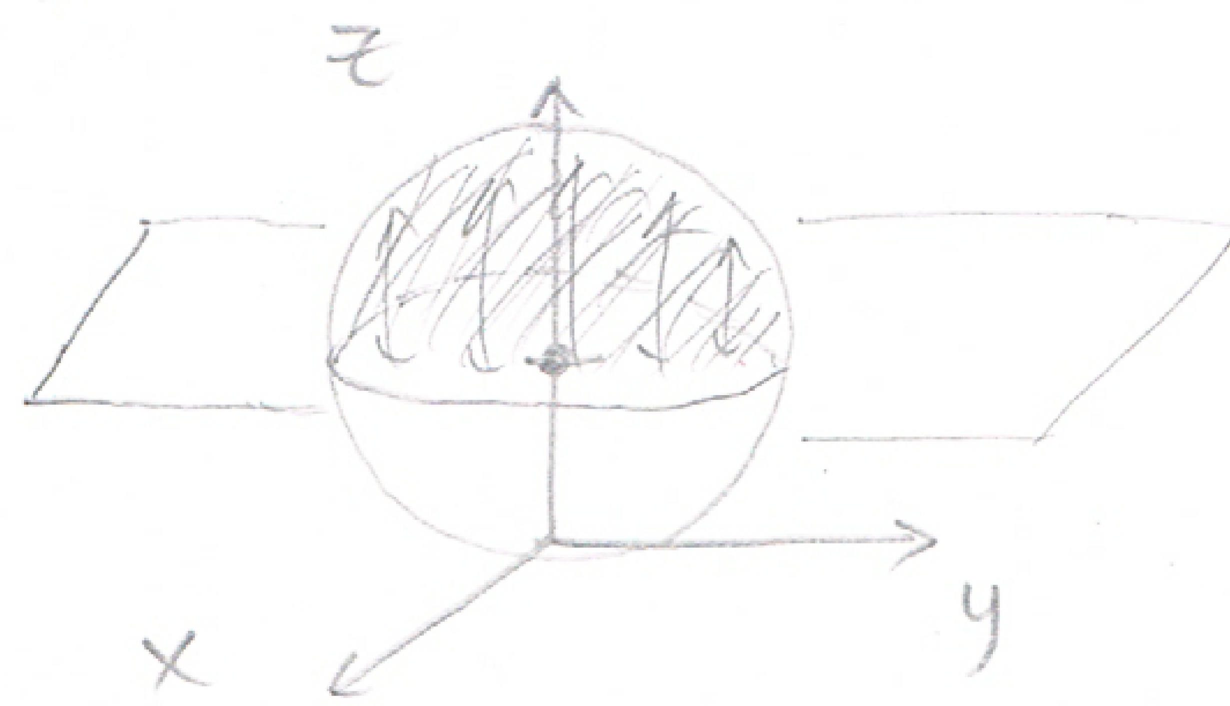
a): Sketch and describe in words the region D which is given in spherical coordinates by the inequalities

$$sec\ \phi \leq \rho \leq 2cos\ \phi \tag{1}$$

(5 points)

- $\sec\phi \le \rho$ is like $1 \le \rho\cos\phi$; $\boxed{1 \le z}$.
- $\rho \le 2\cos\phi$ is like $\rho^2 \le 2\rho\cos\phi$; $x^2+y^2+z^2 \le 2z$

or, complete the square, $\boxed{x^2+y^2+(z-1)^2 \le 1}$



The intersection is what the inequalities describe.

↳ It's the top half of the ball $x^2+y^2+(z-1)^2 \le 1$.

b): Let f, g, h be smooth functions on the real line R. Is the vector field $F(x, y, z) = f(x)\hat{i} + g(y)\hat{j} + h(z)\hat{k}$ conservative? (5 points)

- Partials are all smooth for components ✓ (infinitely differentiable)

- $$\nabla \times \vec{F} = \det \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f(x) & g(y) & h(z) \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial h(z)}{\partial y} - \frac{\partial g(y)}{\partial z}\right) + \hat{j}\left(\frac{\partial f(x)}{\partial z} - \frac{\partial h(z)}{\partial x}\right) + \hat{k}\left(\frac{\partial g(y)}{\partial x} - \frac{\partial f(x)}{\partial y}\right)$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k} = \vec{0}$$

because the functions only have these single variable dependencies!

Since $\vec{F}$'s components are smooth and $\nabla \times \vec{F} = 0$,

$\vec{F}$ is conservative ✓

Q3] Let C be a simple closed positively oriented curve which encloses a region D whose area is equal to 2. Calculate the following line integral

$$\int_C (2x + y^2)dy + (x^2 + y)dx \tag{2}$$

(Hint: Use Green's theorem) (10 points)

we can use because curve is closed ✓

$$\oint_C P\,dx + Q\,dy = \iint_{D_{\text{enclosed}}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$ is Green's Thm.

⚠ Careful! Here, $P = x^2 + y$ and $Q = 2x + y^2$!

so $$\oint_C (2x+y^2)\,dy + (x^2+y)\,dx = \iint_{D_{\text{enclosed}}} \frac{\partial}{\partial x}(2x+y^2) - \frac{\partial}{\partial y}(x^2+y)\,dA$$

$$= \iint_{D_{\text{enclosed}}} (2-1)\,dA$$

$$= \iint_{D_{\text{enclosed}}} dA$$; since $\text{Area}(D) = 2$,

$$= \text{Area}(D) = \boxed{2}$$

Q4] If f is continuous, show that

$$\int_0^x \int_0^y \int_0^z f(t)dtdzdy = \frac{1}{2}\int_0^x (x-t)^2 f(t)dt \tag{3}$$

(Hint: Identify the region E such that $\int_0^x \int_0^y \int_0^z f(t)dtdzdy = \int\int\int_E f dV$ and change the order of integration to $\int\int\int dydzdt$) (10 points)

☆ There's two ways (in my opinion) to do this.

Solution 1 (I prefer this way): Change the orders in pairs (as double integrals) until the desired order is reached.

Here, we go from $dt\,dz\,dy$ to either $dz\,dy\,dt$ or $dy\,dz\,dt$ (dt must be last)

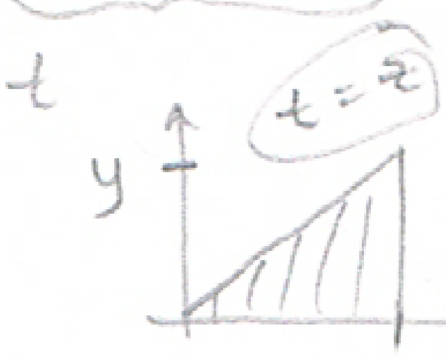
Plan: 1st change $dt\,dz\,dy \to dz\,dt\,dy$,

2nd change $dz\,dt\,dy \to dz\,dy\,dt$.

1st: To switch $dt\,dz$, since we only work with z, t, $\boxed{\text{treat } y \sim \text{constant}}$

so we see $\int_{y=0}^{x} \underbrace{\int_{z=0}^{y} \int_{t=0}^{z}}$ are the bounds, and in (z,t) plane,

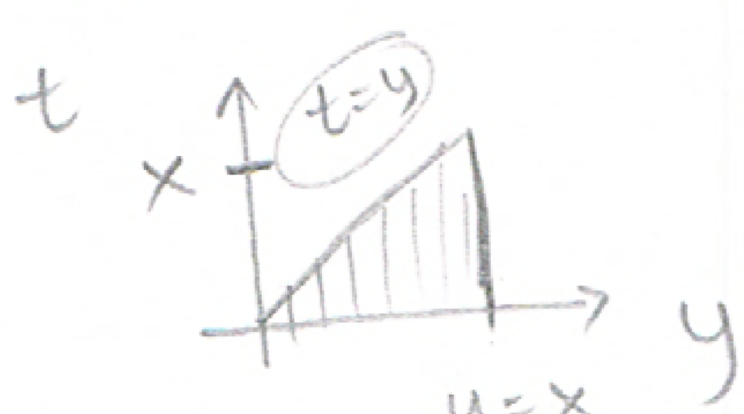
with $y \sim$ constant,



is the domain, so to reorder, we see $\boxed{t \le z \le y \text{ and } 0 \le t \le y}$.

So, we have $\int_{y=0}^{x}\int_{t=0}^{y}\int_{z=t}^{y} f(t)\,dz\,dt\,dy$ as our equivalent integral.

2nd: Take this and now switch $dt\,dy$. Here, in (t,y) plane,



is the domain, so to reorder, we see $\boxed{t \le y \le x,\ 0 \le t \le x}$ we have $\int_{t=0}^{x}\int_{y=t}^{x}\int_{z=t}^{y} f(t)\,dz\,dy\,dt$

is now our integral.

Now, we integrate, $\displaystyle\int_{t=0}^{x}\int_{y=t}^{x}\int_{z=t}^{y} f(t)\,dz\,dy\,dt = \int_{t=0}^{x}\int_{y=t}^{x} f(t)\cdot z\Big|_{z=t}^{y}\,dy\,dt$

$$= \int_{t=0}^{x}\int_{y=t}^{x} f(t)(y-t)\,dy\,dt = \int_{t=0}^{x} f(t)\cdot\frac{(y-t)^2}{2}\Big|_{y=t}^{x}\,dt$$

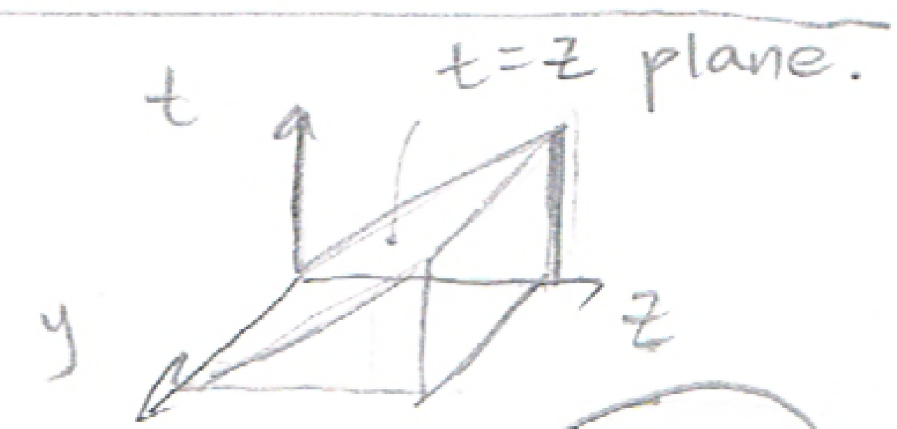
is zero at $y=t$.

$$= \frac{1}{2}\int_{t=0}^{x} f(t)\cdot(x-t)^2\,dt \checkmark$$

we're done ✓

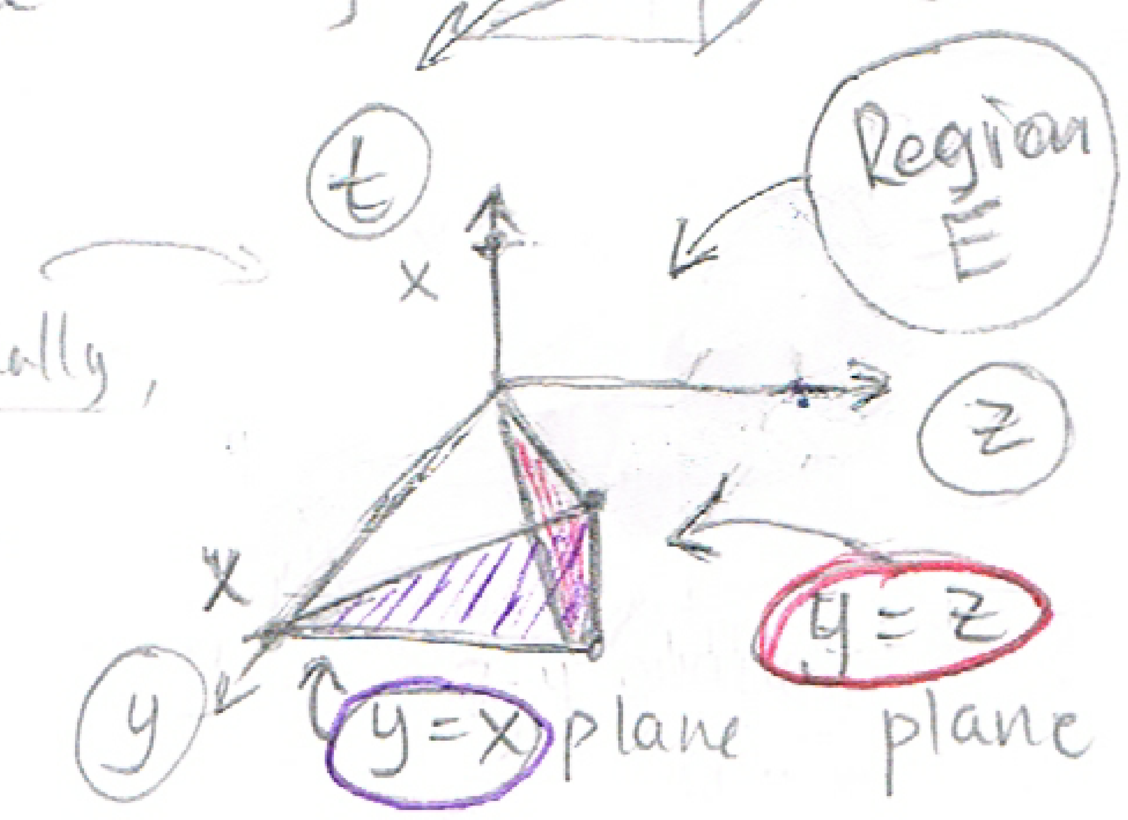
Solution 2: Try to identify E and rewrite.

Since $0 \le t \le z$, we 1st have a wedge like



Then we restrict $0 \le z \le y$, so we essentially cut this diagonally,

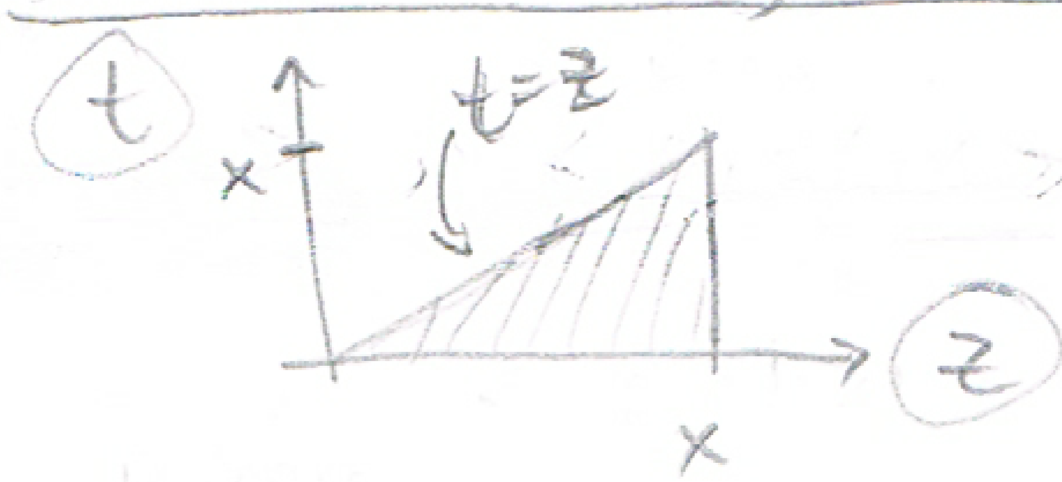
And we stop at some constant x, on y-axis



- Now, when we integrate, dt must be last
- Let us do $dy\,dz\,dt$ order. This way, looking at the figure,

we see $\boxed{z \le y \le x}$ y is between these planes, where recall x is really a constant.

- Then, the $dz\,dt$ shadow is just



so to do $dz\,dt$, $\boxed{\begin{array}{l} t \le z \le x \\ 0 \le t \le x \end{array}}$.

Our Integral is thus $\displaystyle\int_{t=0}^{x}\int_{z=t}^{x}\int_{y=z}^{x} f(t)\,dy\,dz\,dt$

Integrating through dy and dz, $\overset{y\text{-indep}}{=} \displaystyle\int_{t=0}^{x}\int_{z=t}^{x} (x-z)f(t)\,dz\,dt$

(let $u = x-z$, $du = -dz$)

$$= \int_{t=0}^{x} -\frac{(x-z)^2}{2}\Big|_{z=t}^{z=x} f(t)\,dt = \int_{t=0}^{x}\left[0 + +\frac{(x-t)^2}{2}\right] f(t)\,dt$$

$$= \frac{1}{2}\int_{t=0}^{x} (x-t)^2 f(t)\,dt \checkmark$$

Q5] The following question has 2 parts:

a): Let S be the surface given in parametric form by

$$x = u^2 + 4, y = v^3 - 1, z = u + v \tag{4}$$

Show that the tangent plane to S at $(5, 0, 0)$ intersects the y axis at $(0, 15/2, 0)$. (9 points)

1st the plane eqn: Need $\vec{r_u} \times \vec{r_v} \Big|_{At\ (u_0, v_0)}$ for normal.

$\vec{r_u} = \langle 2u, 0, 1 \rangle$

$\vec{r_v} = \langle 0, 3v^2, 1 \rangle$ $\rightsquigarrow$ $\vec{r_u} \times \vec{r_v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2u & 0 & 1 \\ 0 & 3v^2 & 1 \end{vmatrix} = -3v^2\hat{i} - 2u\hat{j} + 6uv^2\hat{k}$.

At $(5,0,0)$, $\Rightarrow$ $u^2+4=5 \rightarrow u = \pm 1$

$v^3 - 1 = 0 \rightarrow$ $\boxed{v = 1}$

$u + v = 0 \rightarrow$ $\boxed{u = -1 \text{ needed}}$ $\Rightarrow \vec{r_u} \times \vec{r_v} \Big|_{(-1,1)} = \langle -3, 2, -6 \rangle$.

Thus, plane eqn: $-3(x-5) + 2y - 6z = 0$.

Thus, the pt of intersection is $(0, \frac{15}{2}, 0)$ ✓

To intersect y-axis, have $x = z = 0$ & solve for y; $-3(-5) + 2y - 0 = 0$,

$y = 15/2$

b): Let f be a scalar function and F be a vector field. Show that $curl(fF) = f curl(F) + \nabla f \times F$. (6 points)

(let $\vec{F} = \langle P, Q, R \rangle$) So $f\vec{F} = \langle fP, fQ, fR \rangle$

$$\nabla \times \langle fP, fQ, fR \rangle = \det \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ fP & fQ & fR \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(fR) - \frac{\partial}{\partial z}(fQ)\right] + \hat{j}\left[\frac{\partial}{\partial z}(fP) - \frac{\partial}{\partial x}(fR)\right] + \hat{k}\left[\frac{\partial}{\partial x}(fQ) - \frac{\partial}{\partial y}(fP)\right]$$

$$= \hat{i}\left[f_y R + f R_y - f_z Q - f Q_z\right] + \hat{j}\left[f_z P + f P_z - f_x R - f R_x\right]$$

$$+ \hat{k}\left[f_x Q + f Q_x - f_y P - f P_y\right]$$

Regroup

$$= \underbrace{\left[f(R_y - Q_z)\hat{i} + f(P_z - R_x)\hat{j} + f(Q_x - P_y)\hat{k}\right]}_{} + \underbrace{\left[(f_y R - \cancel{R_y f}\ f_z Q)\hat{i} + (f_z P - f_x R)\hat{j} + (f_x Q - f_y P)\hat{k}\right]}_{}$$

$$= f(\nabla \times \vec{F}) + \nabla f \times \vec{F}$$

✓

$\frac{9}{9}$

Q6] Use Green's theorem to prove the change of variable formula for a double integral for the case where $f(x, y) = 1$.

$$\int\int_R dxdy = \int\int_S \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv \tag{5}$$

where $R = T(S)$ and $T : \boxed{x = g(u, v), y = h(u, v).}$ (I will show it in class how to do it.) (15 points)

Proof: We can use Green's Thm to get areas! Here, recall:

$$\iint_R dxdy = \iint_R dA \overset{\text{Green's}}{=} \oint_{\partial R} x\,dy$$

where "∂R" denotes the boundary of R.

Now imposing the chg of variables, the boundary changes and we have chain rule on dy,

$$= \int_{\partial S} \underbrace{g(u,v)}_{x} \underbrace{\left(\frac{\partial h(u,v)}{\partial u} du + \frac{\partial h(u,v)}{\partial v} dv\right)}_{dy}$$

☆ Don't forget that x, y are now functions of u and v.

$$= \int_{\partial S} g\frac{\partial h}{\partial u} du + g\frac{\partial h}{\partial v} dV \; ;$$

Now "$P(u,v) = g\frac{\partial h}{\partial u}$" and "$Q(u,v) = g\frac{\partial h}{\partial v}$".

Apply Green's! $=$

$$\iint_S \left(\frac{\partial \text{"}Q\text{"}}{\partial u} - \frac{\partial \text{"}P\text{"}}{\partial v}\right) dudv = \iint_S \frac{\partial}{\partial u}\left(g\frac{\partial h}{\partial v}\right) - \frac{\partial}{\partial v}\left(g\frac{\partial h}{\partial u}\right)$$

$$= \iint_S \left(\frac{\partial g}{\partial u}\frac{\partial h}{\partial v} + \cancel{g\frac{\partial^2 h}{\partial u \partial v}}\right) - \left(\frac{\partial g}{\partial v}\frac{\partial h}{\partial u} + \cancel{g\frac{\partial^2 h}{\partial v \partial u}}\right) dudv$$

cancels! cancels!

$$= \iint_S \left(\frac{\partial g}{\partial u}\frac{\partial h}{\partial v} - \frac{\partial g}{\partial v}\frac{\partial h}{\partial u}\right) dudv$$

Is the Jacobian!

defn! $=$

$$\iint_S \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudV$$

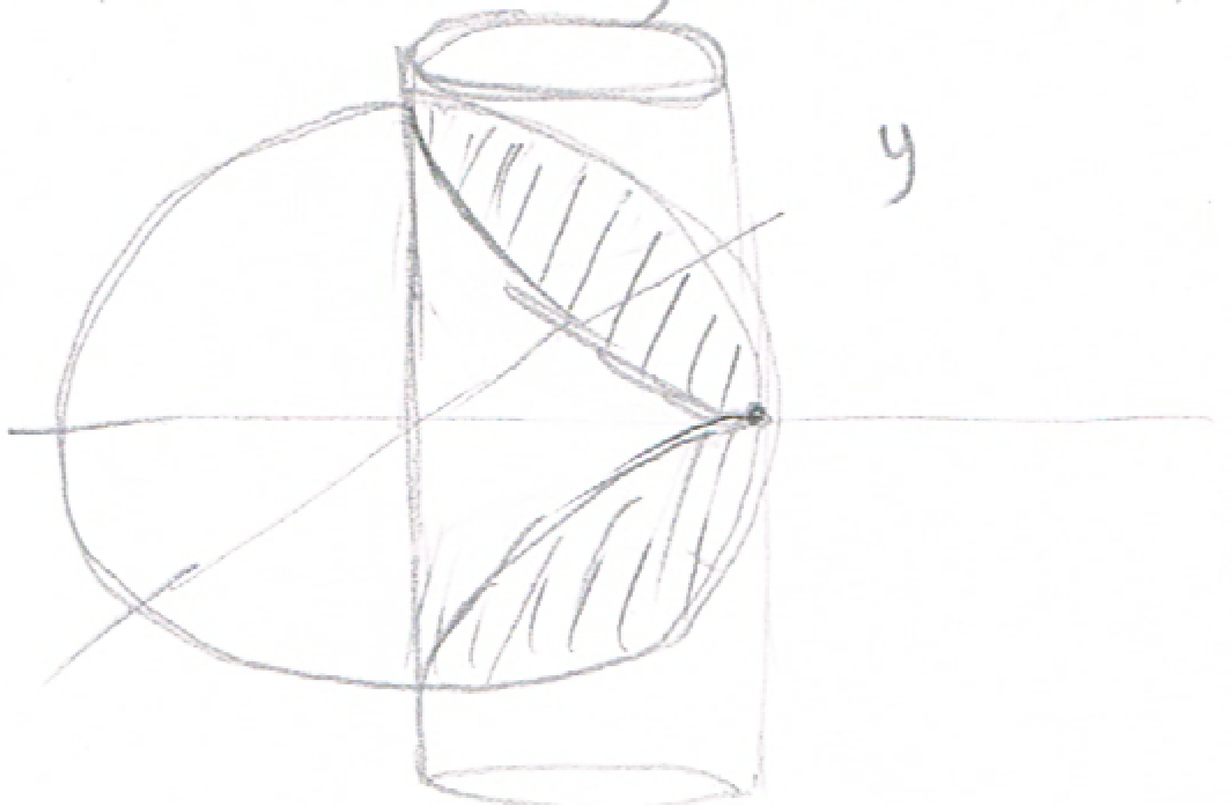
since $x = g(u,v)$, $y = h(u,v)$

▨

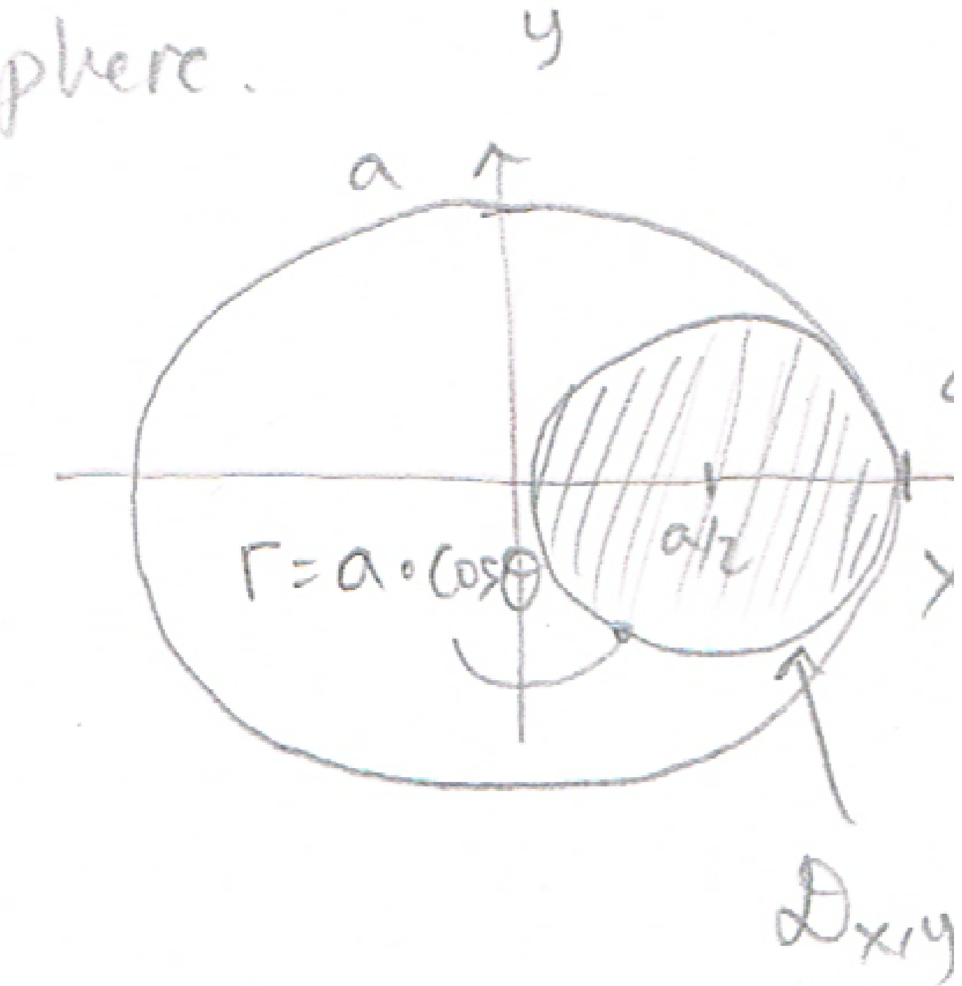
Q7] **Find the area of the part of the sphere $x^2 + y^2 + z^2 = a^2$ that lies inside the cylinder $x^2 + y^2 = ax$. (15 points)**

↳ 1st note $x^2+y^2 = ax \Rightarrow \left(x-\frac{a}{2}\right)^2 + y^2 = \frac{a^2}{4}$ (Radius is $\frac{a}{2}$)

so, its like the cylinder drills out one side of sphere.



Top View



Also, we can just get the area of top half and double it.

☆ View the sphere as graph of $z = \sqrt{a^2-x^2-y^2}$ (for top half)

Then, $\text{Area} = \iint_{D(x,y)} \sqrt{1 + \frac{(-2x)^2+(-2y)^2}{\left(2\sqrt{a^2-x^2-y^2}\right)^2}}\, dx\,dy$

$= \iint_{D_{x,y}} \sqrt{1+\frac{x^2+y^2}{a^2-x^2-y^2}}\, dx\,dy = \iint_{D_{x,y}} \sqrt{\frac{a^2 \cancel{-x^2} \cancel{-y^2} + \cancel{x^2} + \cancel{y^2}}{a^2-x^2-y^2}}\, dx\,dy$

☆ Note the circle is given by $r = a\cdot\cos\theta$

Polar $\stackrel{}{=} \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{r=0}^{a\cdot\cos\theta} \frac{\sqrt{a^2}\cdot r\,dr\,d\theta}{\sqrt{a^2-r^2}}$ $\quad u = a^2 - r^2$, $du = -2r\,dr$

$= \int_{-\pi/2}^{\pi/2} \int \frac{-a\cdot du}{2\sqrt{u}}\, d\theta = \int_{-\pi/2}^{\pi/2} -a\sqrt{u}\,\Big|_{u_0}^{u_f}\, d\theta$

$= \int_{-\pi/2}^{\pi/2} -a\sqrt{a^2-r^2}\,\Big|_{r=0}^{a\cdot\cos\theta}\, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -a\left(\sqrt{a^2(1-\cos^2\theta)} - \left(\sqrt{a^2}\right)\right) d\theta$ $\quad \sqrt{a^2\sin^2\theta}$

Even fn

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(a^2 - a^2\underline{|\sin\theta|}\right) d\theta = 2a^2(\theta + \cos\theta)\Big|_0^{\pi/2} = \boxed{2a^2\left(\frac{\pi}{2} - 1\right)}$

Then, total area is top + bot = double $= \boxed{2a^2(\pi - 2)}$

Q8] The following question has 2 parts.

a): The temperature $u(x,y,z)$ at a point in a ball centered at origin with conductivity K equals $\frac{1}{\sqrt{x^2+y^2+z^2}}$. Find the rate of the heat flow across a sphere of radius a with center at origin. (9 points)

$\cdot\ \nabla u = \frac{\langle -2x, -2y, -2z \rangle}{2(x^2+y^2+z^2)^{3/2}}$ then.

so, $\vec{F} = -K\nabla u = \frac{+K\langle +x, +y, +z \rangle}{(x^2+y^2+z^2)^{3/2}}$. Also, recall

$\vec{n}_{\text{sphere}} = \langle 2x, 2y, 2z \rangle$

$\hookrightarrow \hat{n} = \frac{\langle x, y, z \rangle}{\sqrt{x^2+y^2+z^2}}$

Then, rate $= \iint_S \vec{F} \cdot \vec{dS} = \iint_S \vec{F} \cdot \hat{n}\, dS$

$= \iint_S \frac{+K(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{3/2+1/2}}\, dS \overset{\text{on sphere}}{=} + \iint_S \frac{K(a^2)}{(a^4)}\, dS = \frac{K}{a^2} \iint_S dS$

$= \frac{+K}{a^2} \cdot \text{area (sphere)} = \frac{+K}{\cancel{a^2}} \cdot 4\pi \cancel{a^2}$

$= \boxed{4\pi K}$

b): Suppose F is the inverse square force field, i.e.

$$F(x,y,z) = \frac{\mathbf{r}}{|\mathbf{r}|^3} \tag{6}$$

where $\mathbf{r} = x\hat{i} + y\hat{j} + z\hat{k}$. Find the work done in moving an object from a point P_1 to P_2 along the line segment joining P_1 and P_2 in terms of the distances d_1 and d_2 from these points to the origin. (6 points)

Here, $\vec{F} = \frac{\langle x, y, z \rangle}{(x^2+y^2+z^2)^{3/2}}$ then.

☆ The key is that, from (a), $\vec{F} = \nabla u = \nabla\left(\frac{1}{(x^2+y^2+z^2)^{1/2}}\right)$!!

Mainly, $\vec{F}$ is conservative on any domain that excludes the origin! due to its division by 0 there.

☆ Because conservative, Work $= \int_C \vec{F} \cdot dr = u(\text{final}) - u(\text{initial})$.

So work $= u(P_2) - u(P_1)$. Since $u = \frac{1}{\sqrt{x^2+y^2+z^2}} = \frac{1}{\text{distance}}$,

$= \boxed{\frac{1}{d_2} - \frac{1}{d_1}}$

And this only applies if P_1 and P_2 are NOT the origin!